CALCULUS OF VARIATIONS AND EXTERIOR DIFFERENTIAL SYSTEMS

The aim of the course is to offer an accessible introduction to some of the fundamental connections between differential geometry, Lie groups, and integrable Hamiltonian systems. It will be suitable for students in pure and applied mathematics and other areas such as dynamical systems, control theory, and mathematical physics. The specific goal of the course is to show how to implement the theory of exterior differential systems in one independent variable in the study of constrained variational problems. This method becomes extremely effective for non-degenerate variational problems with symmetries. The theory of exterior differential systems is used both to provide the theoretical background, and to work out specific examples drawn from differential geometry, integrable systems, and mechanics. The subject lies at the confluence of several areas of mathematics, such as differential and symplectic geometry, Lie groups, geometric control theory, and mathematical physics. Because of the richness of concrete examples and explicit solutions to specific problems using general techniques, this course may be especially useful to students beginning their graduate program.

(1) Background : manifolds, tangent and cotangent bundle, differential forms, exterior derivative, vector fields, Lie derivative, contraction, H. Cartan's formula. Frame bundles, Cartan-Darboux, curves in Euclidean space.
(2) Jet Spaces : canonical subbundle of $J^1(\mathbb{R}, M)$, differential ideals, integral manifolds, canonical Pfaffian EDS in $J^1(\mathbb{R}, M)$
(3) Exterior Differential Systems (EDS) : integral manifolds, EDS with constraints.
(4) Variational Problems : variation vectors, extremal solution curves, classical variational problems in $J^1(\mathbb{R}, M)$, mechanical systems, constraints.
(5) Infinitesimal Variations : variations, infinitesimal variations, normal bundle along an integral manifold, first version of the Euler-Lagrange equations.
(6) Euler-Lagrange equations : assumptions required, proof of sufficiency.
(7) Classical Variational Problems : the classical Euler-Lagrange equations. mechanical systems, geodesics.
(8) Riemannian Geometry and Geodesics : Frame bundle on surfaces, structure equations, covariant derivative, geodesic curvature, variation of length.
(9) Elastic Curves in Space Forms Surfaces : Detailed analysis of variation of $\kappa^2/2$ in a surface, constant Gauss curvature case.
(10) Elastic Curves continued : Euclidean case.
(11) The Euler-Lagrange EDS on Momentum Space : Darboux coordinates, Cartan System $C(\Psi)$, integral elements, momentum space associated to (J, ω, ϕ).
(12) Euler-Lagrange EDS for Classical Variational Problems : non-degeneracy, Hamiltonian, mechanical systems.
(13) Good Form, Derived Flag, Cauchy Characteristics : derived flag, Cartan integer, adapted coframe.

(14) Strongly Non-degenerate Variational Problems : Cartan's Lemma, associated quadratic form, $Y = Z_1$ in this case.
(15) Elastic Curves in $\mathbb{R}^3$: Frenet frames, Euclidean group $\mathbb{E}(3)$, variational problem with Lagrangian $L(\kappa)$.
(16) Elastic Curves in $\mathbb{R}^3$ continued : detailed calculations to show $Y = Z_3$, first integrals and solutions when $L(\kappa) = \kappa^2/2$.
(17) First Integrals and Noether's Theorem : infinitesimal symmetry, classical variational problems examples. time-shift invariance, Hamiltonian.
(18) Cyclic Coordinates and Clairault's Theorem : variation of kinetic energy on curves in surfaces of revolution, Maupertuis principle, implications in Clairault's Theorem.
(19) Wheel Rolling on a Plane : the configuration space identified with $J^1(\mathbb{R}, \mathbb{R}^2 \times S^1 \times S^1)$, the constraint equations, kinetic energy.
(20) Wheel Rolling on a Plane continued : momentum space, the Euler-Lagrange EDS, an auxiliary variational problem $(\widetilde{\mathcal{I}}, \omega, \phi)$ and its solution curves.